1.

DISSERTATIO ACADEMICA;

DE MOTU CORPORUM LIBERO IN MEDIO RESISTENTE;

CUJUS PARTEM PRIMAM,

VENIA AMPLISS. FACULTATIS PHILOSOPH.
IN IMPERIALI ACADEMIA ABOËNSI,

PUBLICO EXAMINI MODESTE SUBJICIUNT

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42.

§. I.

Magnum in Scientia motûs locum sibi vindicat theoria motûs *liberi*, i. e. motûs ita comparati, ut nihil impediat, quo minus eam corpus sequatur directionem, juxta quam, tum ratione inertiæ, tum ratione virium sollicitantium, progredi deberet. Hanc igitur theoriam ab omnibus fere Mechanices Auctoribus copiose sane pertractatam deprehendis; nec in re tam nota, aliquid novi facile attuleris. Quæ cum ita sint, ipsa denique methodus, qua hæc proponatur doctrina, magnam, siquando, attentionem mereri videtur; quod idem fere sensisse videntur Auctores, suum quisque modum, in tractanda hac materie, sequentes. Nobis quoque, pro modulo virium, aliquid hac in re tentasse liceat; quod quidem non ita interpretatum volumus, quasi aliorum in his operam quodammodo emendandam censeamus: nobis tantum propositum est, juvenilium in hisce scientiis virium qualecumque fecisse experimentum. Benignæ igitur lectoris censuræ sequentes offerimus pagellas, in quibus motûs quidem liberi, in hypothesi virium sæpissime occurrentium considerati, determinationem, generali quadam propositione complecti conabimur, ex qua deinde solita, quæ apud Auctores passim obveniunt, problemata huc spectantia, tamquam casus speciales, sive corollaria, deducemus. Quo rem quidem eo generalius persequamur, motus omnes in medio quomodocumque resistente considerabimus; sic scilicet hypothesin *vacui* simul comprehendentes, cum in hoc abeat medium quodcumque, posita ejus densitate = o. Corpus adeo motum, in dato quolibet loco, duplicis omnino generis vi sollicitari ponimus, altera velocitatis functione motuique semper directe contrariâ, quæ memorati est medii *resistentia;* alterâ, cujus quantitas atque directio e motu corporis non pendet, quamque hanc ob caussam *absolutæ* nomine insigniemus. In hoc quidem Auctores sequimur: ceterum facile patet, hanc hypothesin pro medio potentiarum vi sollicitato, ideo revera non satis esse accuratam, quod pressio illa medii, quâ hoc in casu, præter resistentiam, urgetur corpus (quamque, in determinanda vi absoluta, spectari oporteret), ex motu corporis quodammodo

dammodo pendeat. Hac vero consideratione calculi implicandi non fuerunt. Ceterum, id quoque notandum, inter plures, quas expressio continet resistentiæ, quantitates, nos heic tantum *velocitatem* mediique *densitatem*, progrediente corpore, variationi obnoxias, sistere. Superficiem igitur projectilis ipsi motus plagæ adversam, tenacitatem medii, & sic porro, in diversis trajectoriæ locis non diversas sed omnino constantes, assumimus. Hisce observatis, ipsam denique rem adgrediamur, quam ita quidem proponemus, ut primo motus in spatio, vi problematis universalis, breviter consideremus, dein vero ex propositione generali specialiorem, quæ pro motibus quibuscumque liberis in plano valeat, eliciendo, ad motus denique planos propius considerandos, progrediamur. Propositio vero universalis, cui, tamquam fundamento, omnia in hac opella nituntur, in sequente §:o proponitur.

§. II.

Sit K (vide fig.) punctum datum in medio quocumque dato, KO data positione recta, sitque corpus datum, data cum velocitate, secundum KO projectum, in quod progrediens, præter medii resistentiam, tres quæcumque agant datæ potentiæ absolutæ, secundum datas tres directiones constantes sibi invicem normales, sollicitantes. Determinanda sunt omnia, quæ ad motum *spectant corporis projecti; i. e. inveniatur natura trajectoriæ descriptæ KDF, velocitas in puncto quolibet D, tempusque, quo data quædam lineæ KDF pars describatur.*

Ad generale hocce problema solvendum, ducatur per punctum K planum ABC, cuicumque directionum normalium perpendiculare, datumque sic angulum cum data KO efficiens. Demissâ jam, a puncto quolibet trajectoriæ D, DC in planum ABC normali, ductâque per C, ubi DC plano ABC occurrit, CB alteri cuidam directionum normalium parallela, ducatur denique, per punctum quodlibet ipsius CB, recta in sequentibus fixa AB, in plano ABC ipsi CB ad normam: erit AB tertiæ directionum normalium parallela. Potentiarum absolutarum, secundum directiones normales agentium, eam esse indolem, ut ab A B versus, a B C versus, a C D versus, agant, statuere possumus. Sumto jam, in linea AB, puncto quolibet fixo A, sint: $AB = x$, $BC = y$, $CD = z$, velocitas in D (F versus) $= v$, densitas medii in $D = D$, resistentiæ vis in D (solito modo, per celeritatem tempore dato 1 corpori dato uniformiter conciliandam, expressa) $= R = \varphi\ (D,\ v)$, potentiæque absolutæ, in puncto D, secundum AB, BC, CD, agentes (eâdemque, quâ nuper resistentia, mensurâ æstimatæ) $= L$, M, N. Ulterius, a puncto trajectoriæ

ctoriæ F, ipsi D infinite propinquo, ducatur FE ipsi DC parallela, planoque ABC occurrens in E, perque E transeat recta EG ipsi CB parallela, lineæ abscissarum ABG occurrens in G. Juncta CE, ducatur, per D, DP ipsi CE parallela, perque C, CQ ipsi ABG æquidistans. Sit jam $BG = \mathrm{d}x$, $EQ = \mathrm{d}y$, $FP = \mathrm{d}z$, $DP = CE = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \mathrm{d}q$, $DF = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2} = \mathrm{d}s$, tempusque, quo spatiolum $\mathrm{d}s$, velocitate v, percurritur $= \mathrm{d}t$. Ut jam omnia, quantum fieri potest, generaliter proponamus, nullam differentialium, in hac propositione occurrentium, constantem accipiemus; hoc vero effecturis sequenti modo operandum. Quia:

Resistentiæ in D vis secundum DC - - - $= \frac{R\mathrm{d}z}{\mathrm{d}s}$,

— — — — — — EC - - - $= \frac{R\mathrm{d}q}{\mathrm{d}s}$,

— — — — — — $CB = \frac{R\mathrm{d}q}{\mathrm{d}s} \cdot \frac{\mathrm{d}y}{\mathrm{d}q} = \frac{R\mathrm{d}y}{\mathrm{d}s}$,

— — — — — — $BA = \frac{R\mathrm{d}q}{\mathrm{d}s} \cdot \frac{\mathrm{d}x}{\mathrm{d}q} = \frac{R\mathrm{d}x}{\mathrm{d}s}$;

erit utique:

Tota, secundum AB, vis $= L - \frac{R\mathrm{d}x}{\mathrm{d}s}$,

Tota, secundum BC, vis $= M - \frac{R\mathrm{d}y}{ds}$,

Tota

Tota, secundum CD, vis $= N - \frac{Rdz}{ds}$.

At vero, erit quoque:

Velocitas in D secundum CD - - - - - - $= \frac{vdz}{ds}$,

— — — — — — CE - - - - - - $= \frac{vdq}{ds}$,

— — — — — — $BC = \frac{vdq}{ds} \cdot \frac{dy}{dq} = \frac{vdy}{ds}$,

— — — — — — $AB = \frac{vdq}{ds} \cdot \frac{dx}{dq} = \frac{vdx}{ds}$.

Habemus igitur, per notas virium accelerantium formulas, æquationes:

$$\frac{\left(\frac{vdx}{ds}\right) . d\left(\frac{vdx}{ds}\right)}{dx} = L - \frac{Rdx}{ds} \ldots\ldots (1),$$

$$\frac{\left(\frac{vdy}{ds}\right) . d\left(\frac{vdy}{ds}\right)}{dy} = M - \frac{Rdy}{ds} \ldots\ldots (2),$$

$$\frac{\left(\frac{vdz}{ds}\right) . d\left(\frac{vdz}{ds}\right)}{dz} = N - \frac{Rdz}{ds} \ldots\ldots (3);$$

quæ ea sunt principia, quæ ad nostri solutionem problematis abunde sufficiant. Ex his quippe æquaquationibus valores ipsorum v & R eliciendo, ad duas inter x, y, z æquationes differentiales perveniemus, quæ naturam trajectoriæ quæsitæ KDF definient. Quod si indicatas igitur differentiationes perficimus, prodibit:

$$\frac{v\left(\mathrm{d}s\,.\,\mathrm{d}v + v\mathrm{d}s\,.\,\frac{\mathrm{d}^2x}{\mathrm{d}x} - v\,.\,\mathrm{d}^2s\right)}{\mathrm{d}s^3} = \frac{L}{\mathrm{d}x} - \frac{R}{\mathrm{d}s} \ldots (4),$$

$$\frac{v\left(\mathrm{d}s\,.\,\mathrm{d}v + v\mathrm{d}s\,.\,\frac{\mathrm{d}^2y}{\mathrm{d}y} - v\,.\,\mathrm{d}^2s\right)}{\mathrm{d}s^3} = \frac{M}{\mathrm{d}y} - \frac{R}{\mathrm{d}s} \ldots (5),$$

$$\frac{v\left(\mathrm{d}s\,.\,\mathrm{d}v + v\mathrm{d}s\,.\,\frac{\mathrm{d}^2z}{\mathrm{d}z} - v\,.\,\mathrm{d}^2s\right)}{\mathrm{d}s^3} = \frac{N}{\mathrm{d}z} - \frac{R}{\mathrm{d}s} \ldots (6).$$

Subtracta jam æquatione (5) a (4), habebitur:

$$\frac{v^2\mathrm{d}s\,.\,\frac{\mathrm{d}^2x}{\mathrm{d}x}}{\mathrm{d}s^3} - \frac{v^2\mathrm{d}s\,.\,\frac{\mathrm{d}^2y}{\mathrm{d}y}}{\mathrm{d}s^3} = \frac{L}{\mathrm{d}x} - \frac{M}{\mathrm{d}y}, \text{ i. e.}$$

$$v^2 = \frac{(L\mathrm{d}y - M\mathrm{d}x)\,.\,\mathrm{d}s^2}{\mathrm{d}y\,.\,\mathrm{d}^2x - \mathrm{d}x\,.\,\mathrm{d}^2y} \quad .\ .\ .\ .\ .\ . (7).$$

Pari modo, subtracta æquatione (6) a (4), prodit:

$v^2 =$

$$v^2 = \frac{(Ldz - Ndx) . ds^2}{dz . d^2x - dx . d^2z};$$

nec non, subtracta æquatione (6) a (5):

$$v^2 = \frac{(Mdz - Ndy) . ds^2}{dz . d^2y - dy . d^2z}.$$

Hosce tres valores ipsius v^2 inter se comparando, tres prodeunt æquationes coordinatas x, y, z continentes, quæ tamen non nisi unam rem exprimunt; omnesque identicæ sunt huic:

$$L(dz . d^2y - dy . d^2z) + M(dx . d^2z - dz . d^2x) + N(dy . d^2x - dx . d^2y) = 0 \quad \ldots \ldots \quad (I).$$

Porro, si substituatur in æquatione (4) valor ipsius v^2, ex æquatione (7) depromtus, fiatque, per æquationem (I), debita reductio, prodibit:

$$R = \frac{Ldx + Mdy + Ndz}{ds} - \frac{d . \left\{\frac{(Ldy - Mdx) . ds^2}{dy . d^2x - dx . d^2y}\right\}}{2ds};$$

unde, observando quod $R = \varphi(D, v)$, habebitur:

$$\varphi\left(D, ds\sqrt{\frac{Ldy - Mdx}{dy . d^2x - dx . d^2y}}\right) = \frac{Ldx + Mdy + Ndz}{ds} - \frac{d . \left\{\frac{(Ldy - Mdx) ds^2}{dy . d^2x - dx . d^2y}\right\}}{2ds} \quad \ldots \ldots \quad (II).$$

Po-

Positis jam D, L, M, N datis quibuscumque ipsorum x, y, z functionibus, æquationes (I) & (II) quantitates curvam descriptam spectantes tantum continebunt, naturamque sic quæsitæ trajectoriæ definient.

Inventa sic, per integrationem æquationum (I) & (II), indole trajectoriæ KDF, velocitas v, tempusque t, facili negotio investigantur. Habemus enim, per præcedentia, formulas generales:

$$v^2 = \frac{(L\mathrm{d}y - M\mathrm{d}x)\,.\,\mathrm{d}s^2}{\mathrm{d}y\,.\,\mathrm{d}^2x - \mathrm{d}x\,.\,\mathrm{d}^2y},$$

$$\mathrm{d}t^2\left(=\frac{\mathrm{d}s^2}{v^2}\right) = \frac{\mathrm{d}y\,.\,\mathrm{d}^2x - \mathrm{d}x\,.\,\mathrm{d}^2y}{L\mathrm{d}y - M\mathrm{d}x};$$

quamobrem, si, ope æquationum trajectoriæ supra inventarum, habeatur ex. gr. $L = X$, $M = X_1$, $\mathrm{d}y = X_2\,\mathrm{d}x$ (unde $\mathrm{d}y\,.\,\mathrm{d}^2x - \mathrm{d}x\,.\,\mathrm{d}^2y = -\,\mathrm{d}x^2\,.\,\mathrm{d}X_2 = -\,X_3\,\mathrm{d}x^3$), $\mathrm{d}s = X_4\,\mathrm{d}x$, ubi X, X_1, &c. functiones sunt ipsius x; erit utique:

$$v^2 = \frac{(X\,.\,X_2 - X_1)\,.\,X_4^2}{-\,X_3};$$

$$t = \int \mathrm{d}x \sqrt{\frac{-\,X_3}{X\,.\,X_2 - X_1}} + Const.;$$

quos quidem valores differentialibus exemtos vides.

His jam, quantum fieri licuit, solutum est problema nostrum universale, quod quidem adeo late patere facile videtur, ut nullum non casum motûs liberi, per vim resistentiæ viresque absolutas determinati, complectatur. Ad tres enim potentias L, M, N, sibi invicem normales, viribus resolvendis, quamcumque revocari posse vim absolutam, facile patet. Omnia igitur in præsenti opella infra proferenda, corollariorum instar, ex propositione præcedente profluentium, merito habenda sunt. Ceterum hoc loco observandum, plura quidem proponi posse problemata generalia præcedenti reciproca, ut ex. gr. datis curva, velocitate in punctis singulis, densitate medii formaque functionis φ, invenire L, M, N, & sic porro; his vero in præsenti opella, qua e directa motus determinatione, tamquam principio, progredimur, immorari non est visum. Præterea, parum sane commodi hinc reportaremus, cum problemata inversa, ut in sequentibus quoque apparebit, in casu quolibet speciali, ope problematis nostri universalis tractando, eâdem fere facilitate resolvantur, ac si generalia his quoque principia directe adcommodarentur. Ad æquationes igitur (I) & (II), quas solutum dedit problema generale, diligentius considerandas, jamjam progrediamur.

§. III.

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Antequam vero ad alia attendimus, unam alteramve, de indole harum æquationum, observationem attulisse, a re forsitan non erit alienum. Primo quidem intuitu patet, æquationem (I) ex ipsius medii, in quo motus fiat, naturâ, nihil omnino pendere, eamdemque prorsus mansuram, si vel in vacuo moveretur corpus; effectum vero medii, ad mutandum corporis motum, sola indicari æquatione (II). Cum trajectoria igitur a corpore descripta, quasi intersectio duarum superficierum considerari possit, quarum altera æquatione (I) definitur, altera vero æquatione (II); sequitur corpus, in quocumque medio motum, datis ipsis L, M, N, datam quoque quamdam superficiem, ex ipsius medii natura minime pendentem, semper describere; naturamque hujus superficiei, integratione æquationis (I), innotescere. Nos quidem, cum variis, quæ huc spectarent, disquisitionibus immorari non possimus, exemplorum loco, unum tantum vel alterum indicabimus casum, quo, corpore utcumque projecto, *planam* fieri memoratam superficiem, æquatione nostra (I) facile evincitur.

Sint igitur potentiæ L, M, N in datâ inter se ratione, i. e. $M = aL$, $N = bL$; hoc in casu motum in plano necessario fieri, demonstrari facile potest. Erit enim:

 $L(\mathrm{d}z.\mathrm{d}^2y$

$L(\mathrm{d}z \,.\, \mathrm{d}^2 y - \mathrm{d}y \,.\, \mathrm{d}^2 z) + aL(\mathrm{d}x \,.\, \mathrm{d}^2 z - \mathrm{d}z \,.\, \mathrm{d}^2 x) + bL(\mathrm{d}y \,.\, \mathrm{d}^2 x - \mathrm{d}x \,.\, \mathrm{d}^2 y) = o$, i. e.

$\mathrm{d}z \,.\, \mathrm{d}^2 y - \mathrm{d}y \,.\, \mathrm{d}^2 z + a(\mathrm{d}x \,.\, \mathrm{d}^2 z - \mathrm{d}z \,.\, \mathrm{d}^2 x) + b(\mathrm{d}y \,.\, \mathrm{d}^2 x - \mathrm{d}x \,.\, \mathrm{d}^2 y) = o.$

Multiplicatâ hac æquatione per factorem $\left(\frac{1}{(b\mathrm{d}x - \mathrm{d}z)^2}\right)$, integrabilis evadit; fietque, per integrationem,

$\frac{a\mathrm{d}x - \mathrm{d}y}{b\mathrm{d}x - \mathrm{d}z} = C$, i. e. $(a - bC)\,\mathrm{d}x - \mathrm{d}y + C\mathrm{d}z = o$;

iterumque integrando:

$$(a - bC) \,.\, x - y + Cz - C' = o;$$

æquatio ad planum.

Sit $a = o$ vel $b = o$, i. e. evanescat M vel N; habebitur:

$$bCx + y - Cz + C' = o, \text{ vel } ax - y + Cz - C' = o.$$

Quod si autem $a = o$ atque $b = o$, i. e. fiant ambæ M & N æquales nihilo; erit:

$$y - Cz + C' = o,$$

quæ æquatio est plani, ad planum ipsorum y & z normalis.

Ulterius, sit: $L = - cx$, $M = - cy$, $N = - cz$, i. e. tra-

trahatur corpus utcumque projectum perpetuo ad initium abscissarum A. In hoc quoque casu, motum in plano fore, æquatione nostra (I) satis indicatur. Habemus enim:

$$-cx(\mathrm{d}z\,.\,\mathrm{d}^2y-\mathrm{d}y\,.\,\mathrm{d}^2z)-cy(\mathrm{d}x\,.\,\mathrm{d}^2z-\mathrm{d}z\,.\,\mathrm{d}^2x)$$
$$-cz(\mathrm{d}y\,.\,\mathrm{d}^2x-\mathrm{d}x\,.\,\mathrm{d}^2y)=o,\ \text{i. e.}$$

$$x(\mathrm{d}z\,.\,\mathrm{d}^2y-\mathrm{d}y\,.\,\mathrm{d}^2z)+y(\mathrm{d}x\,.\,\mathrm{d}^2z-\mathrm{d}z\,.\,\mathrm{d}^2x)$$
$$+z(\mathrm{d}y\,.\,\mathrm{d}^2x-\mathrm{d}x\,.\,\mathrm{d}^2y)=o;$$

quæ quidem æquatio, per factorem $\left(\frac{z}{(z\mathrm{d}y-y\mathrm{d}z)}\right)$ multiplicata, integrari potest, datque:

$$\frac{z\mathrm{d}x-x\mathrm{d}z}{z\mathrm{d}y-y\mathrm{d}z}=C,\ \text{i. e. } z\mathrm{d}x-x\mathrm{d}z-C(z\mathrm{d}y-y\mathrm{d}z)=o.$$

Multiplicetur hæc æquatio per $\left(\frac{I}{z^2}\right)$; integratione iterum facta, prodibit:

$$\frac{x-Cy}{z}=C',\ \text{i. e. } x-Cy-C'z=o;$$

æquatio planum definiens.

Sed de his jam satis.

§. IV.

Cum æquationes (I) & (II) adeo sint complicatæ, ut in genere tractari nequeant, casus tantum par-

particulares jamjam nobis contemplandi. Hos, pro angustia opellæ, nos optime tractaturos credimus, si exemplo quodam, vim methodi, in propositione generali traditæ, quadantenus illustrante, proposito, insigniores quosdam æquationum (I) & (II) casus, dein breviter examinemus. Proponatur ex. gr. sequens problema, methodo supra allata solvendum:

In medio quocumque dato, duæ sollicitent datæ potentiæ absolutæ P & Q, quarum illa ad punctum tendat immobile A, hæc vero in planum datum ABC, per hoc punctum transiens, normaliter agat; quæritur motus corporis dati, utcumque projecti?

Brevitatis gratia ponendo: $\sqrt{x^2 + y^2 + z^2} = n$, patet hoc in casu haberi:

$$L = -\frac{Px}{n},\ M = -\frac{Py}{n},\ N = -\frac{Pz}{n} - Q;$$

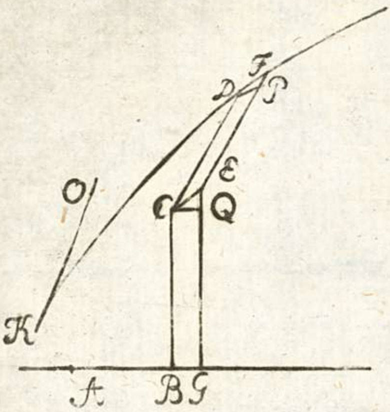
hos igitur valores in æquationibus (I) & (II) substituendo (positâ brevitatis ergo $dx = const.$), prodit:

$$Qndx . d^2y = [(xdz - zdx)d^2y - (xdy - ydx)d^2z] . P \ldots (8),$$

$$\varphi\left(D, ds\sqrt{\frac{(xdy - ydx)P}{ndx . d^2y}}\right) = \frac{-P(xdx + ydy + zdz) - Qndz}{nds}$$

$$\frac{-d . \left\{\frac{(xdy - ydx)\,Pds^2}{ndx . d^2y}\right\}}{2ds} \quad \ldots\ldots\ldots (9);$$

h. e.,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